Решения на задачи за изпит по ОТО

Камен Димитров Младенов, FN

DATE

1. Решение Собствения часовник в собствената координатна система винаги върви най-бързо, защото отнема време на светлината от единия брат да достигне другия брат. Всеки брат изминава еднакво разстояние с еднаква скорост, значи за всеки брат е изминало еднакво количество време.

2. Решение Смятало се е, че светлината се движи през някаква невидима среда, наречена "етер". Ако етерът съществувал, то Земята ще се движи през него и би пораждала етеров вятър. Светлинен лъч ще се движи по-бързо по посоката на вятъра, по-бавно срещу и нещо средно перпендикулярно.

През 1881г. Майкелсон провежда първи "опит" на експеримента, чрез изобретения си "Майкелсонов интерферометър".

Един светлинен източник е поставен пред центъра на полу-прозрачно огледало, наклонено под 45° ъгъл (спрямо светлинния източник). Светлиния сноп се разделя на две половини от предното лице на огледалото: едната се отразява, докато другата преминава. Първата се отразява обратно от подвижно огледало, докато втората се отразява обратно от фиксирано огледало. Полу-прозрачното огледало разделя и връщащите се лъчи на две, като остават две четвъртинки снопове от източника, които "излизат" от задното лице на огледалото.

Резултата е, че спрямо различия в скоростите на двете, произлезли от препендикулярни лъчи, четвъртинки снопове се получава интерференчно изображение, от което може тези различията да се измерят. Обаче, грешката в прецизността на уреда е била твърде голяма, за да потвърди или опровергае етерът.

През 1887г. Майкелсон и Морли подобряват този апарт, като добавят още огледала, през които двете половинки да се отразяват, преди обратната комбинация, така че различията в скоростите да са по-явни. Допълнително, апаратът е бил поставен в подземна затворена стая, върху пясъчник, който сам по себе си е плавал върху живак.

Въртейки апаратът (с много малко съпротивление, благодарение на живакът) и правейки измервания, Майкелсон и Морли намират отрицателен резултат: изчислените разлики са били значително по-малки от теоретическото очакване, за да съществува етър.

3. Решение $00800028 \; mod \; 3 = 0$, тоест ще докажем, че ковариантната производна на тензор от тип $(0,2)$ е тензор от тип $(0,3)$.

$$\nabla g = \nabla_k \, g_{ab} \, \mathrm{d}x^a \mathrm{d}x^b \mathrm{d}x^k = \left(\frac{\partial g_{ab}}{\partial x^k} - \Gamma^m{}_{ak} \, g_{mb} - \Gamma^m{}_{kb} \, g_{ma} \right) \mathrm{d}x^a \mathrm{d}x^b \mathrm{d}x^k$$

Правим смяна на променлива, от x към y, което означава че:

$$\tilde{\Gamma}^i{}_{jk} = \Gamma^a{}_{bc} \frac{\partial y^i}{\partial x^a} \frac{\partial x^b}{\partial y^j} \frac{\partial x^c}{\partial y^k} + \frac{\partial y^i}{\partial x^p} \frac{\partial^2 x^p}{\partial y^j \partial y^k} \implies \Gamma^a{}_{bc} = \left(\tilde{\Gamma}^i{}_{jk} - \frac{\partial y^i}{\partial x^p} \frac{\partial^2 x^p}{\partial y^j \partial y^k} \right) \frac{\partial x^a}{\partial y^i} \frac{\partial y^j}{\partial x^b} \frac{\partial y^k}{\partial x^c} \qquad \mathrm{d}x^a = \frac{\partial x^a}{\partial y^i} \mathrm{d}y^i$$

$$g = g_{ab} \mathrm{d}x^a \mathrm{d}x^b = g_{ab} \frac{\partial x^a}{\partial y^i} \mathrm{d}y^i \frac{\partial x^b}{\partial y^j} \mathrm{d}y^j = g_{ab} \frac{\partial x^a}{\partial y^i} \frac{\partial x^b}{\partial y^j} \mathrm{d}y^i \mathrm{d}y^j = \tilde{g}_{ij} \mathrm{d}y^i \mathrm{d}y^j \implies g_{ab} = \tilde{g}_{ij} \frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^b}$$

Заедно с $\dfrac{\partial}{\partial x^i} = \dfrac{\partial}{\partial y^a}\dfrac{\partial y^a}{\partial x^i}$ следва че:

$$\frac{\partial g_{ab}}{\partial x^k} = \frac{\partial y^l}{\partial x^k}\frac{\partial}{\partial y^l}\left(\tilde{g}_{ij}\frac{\partial y^i}{\partial x^a}\frac{\partial y^j}{\partial x^b}\right) = \frac{\partial y^l}{\partial x^k}\left(\tilde{g}_{ij}\frac{\partial}{\partial y^l}\left(\frac{\partial y^i}{\partial x^a}\frac{\partial y^j}{\partial x^b}\right) + \frac{\partial y^i}{\partial x^a}\frac{\partial y^j}{\partial x^b}\frac{\partial}{\partial y^l}\left(\tilde{g}_{ij}\right)\right)$$
$$= \frac{\partial y^l}{\partial x^k}\left(\tilde{g}_{ij}\left(\frac{\partial y^i}{\partial x^a}\frac{\partial}{\partial y^l}\left(\frac{\partial y^j}{\partial x^b}\right) + \frac{\partial y^j}{\partial x^b}\frac{\partial}{\partial y^l}\left(\frac{\partial y^i}{\partial x^a}\right)\right) + \frac{\partial y^i}{\partial x^a}\frac{\partial y^j}{\partial x^b}\frac{\partial}{\partial y^l}\left(\tilde{g}_{ij}\right)\right)$$
$$= \frac{\partial y^l}{\partial x^k}\left(\tilde{g}_{ij}\left(\frac{\partial y^i}{\partial x^a}\frac{\partial^2 y^j}{\partial y^l \partial x^b} + \frac{\partial y^j}{\partial x^b}\frac{\partial^2 y^i}{\partial y^l \partial x^a}\right) + \frac{\partial y^i}{\partial x^a}\frac{\partial y^j}{\partial x^b}\frac{\partial \tilde{g}_{ij}}{\partial y^l}\right)$$
$$= \frac{\partial y^l}{\partial x^k}\frac{\partial y^i}{\partial x^a}\frac{\partial^2 y^j}{\partial y^l \partial x^b}\tilde{g}_{ij} + \frac{\partial y^l}{\partial x^k}\frac{\partial y^j}{\partial x^b}\frac{\partial^2 y^i}{\partial y^l \partial x^a}\tilde{g}_{ij} + \frac{\partial y^l}{\partial x^k}\frac{\partial y^i}{\partial x^a}\frac{\partial y^j}{\partial x^b}\frac{\partial \tilde{g}_{ij}}{\partial y^l}$$

Тоест:

$$\begin{aligned}\nabla g = \Bigg(& \frac{\partial y^l}{\partial x^k}\frac{\partial y^i}{\partial x^a}\frac{\partial^2 y^j}{\partial y^l \partial x^b}\tilde{g}_{ij} + \frac{\partial y^l}{\partial x^k}\frac{\partial y^j}{\partial x^b}\frac{\partial^2 y^i}{\partial y^l \partial x^a}\tilde{g}_{ij} + \frac{\partial y^l}{\partial x^k}\frac{\partial y^i}{\partial x^a}\frac{\partial y^j}{\partial x^b}\frac{\partial \tilde{g}_{ij}}{\partial y^l} \\ & - \left(\tilde{\Gamma}^n{}_{il} - \frac{\partial y^n}{\partial x^p}\frac{\partial^2 x^p}{\partial y^i \partial y^l}\right)\cancel{\frac{\partial x^m}{\partial y^n}}\frac{\partial y^i}{\partial x^a}\frac{\partial y^l}{\partial x^k}\tilde{g}_{nj}\cancel{\frac{\partial y^n}{\partial x^m}}\frac{\partial y^j}{\partial x^b} \\ & - \left(\tilde{\Gamma}^n{}_{lj} - \frac{\partial y^n}{\partial x^p}\frac{\partial^2 x^p}{\partial y^l \partial y^j}\right)\cancel{\frac{\partial x^m}{\partial y^n}}\frac{\partial y^l}{\partial x^k}\frac{\partial y^j}{\partial x^b}\tilde{g}_{ni}\cancel{\frac{\partial y^n}{\partial x^m}}\frac{\partial y^i}{\partial x^a} \\ & \Bigg)\frac{\partial x^a}{\partial y^i}\mathrm{d}y^i\frac{\partial x^b}{\partial y^j}\mathrm{d}y^j\frac{\partial x^k}{\partial y^l}\mathrm{d}y^l\end{aligned}$$

$$\tilde{g}_{nj} = \tilde{g}_{ij}\frac{\partial y^i}{\partial y^n} \implies \frac{\partial y^n}{\partial x^p}\frac{\partial^2 x^p}{\cancel{\partial y^i}\partial y^l}\cancel{\frac{\partial y^i}{\partial x^a}}\frac{\partial y^l}{\partial x^k}\tilde{g}_{nj}\frac{\partial y^j}{\partial x^b} = \cancel{\frac{\partial y^n}{\partial x^p}}\frac{\partial^2 x^p}{\partial x^a \partial y^l}\tilde{g}_{ij}\frac{\partial y^i}{\partial x^k}\frac{\partial y^l}{\partial x^k}\frac{\partial y^j}{\partial x^b}$$
$$= \frac{\partial y^i}{\partial x^p}\frac{\partial}{\partial x^a}\left(\frac{\partial x^p}{\partial y^l}\right)\tilde{g}_{ij}\frac{\partial y^l}{\partial x^k}\frac{\partial y^j}{\partial x^b} = \left(\frac{\partial}{\partial x^a}\left(\frac{\partial y^i}{\cancel{\partial x^p}}\frac{\cancel{\partial x^p}}{\partial y^l}\right) - \frac{\partial x^p}{\partial y^l}\frac{\partial}{\partial x^a}\left(\frac{\partial y^i}{\partial x^p}\right)\right)\tilde{g}_{ij}\frac{\partial y^l}{\partial x^k}\frac{\partial y^j}{\partial x^b}$$
$$= 0 - \frac{\cancel{\partial x^p}}{\partial y^l}\frac{\partial^2 y^i}{\partial x^a \cancel{\partial x^p}}\tilde{g}_{ij}\frac{\partial y^l}{\partial x^k}\frac{\partial y^j}{\partial x^b} = -\frac{\partial y^l}{\partial x^k}\frac{\partial y^j}{\partial x^b}\frac{\partial^2 y^i}{\partial y^l \partial x^a}\tilde{g}_{ij}$$
Аналогично $\dfrac{\partial y^n}{\partial x^p}\dfrac{\partial^2 x^p}{\partial y^l \partial y^j}\dfrac{\partial y^l}{\partial x^k}\dfrac{\partial y^j}{\partial x^b}\tilde{g}_{ni}\dfrac{\partial y^i}{\partial x^a} = -\dfrac{\partial y^l}{\partial x^k}\dfrac{\partial y^i}{\partial x^a}\dfrac{\partial^2 y^j}{\partial y^l \partial x^b}\tilde{g}_{ij}$

Следователно:

$$\begin{aligned}\nabla g = \Bigg(& \cancel{\frac{\partial y^l}{\partial x^k}\frac{\partial y^i}{\partial x^a}\frac{\partial^2 y^j}{\partial y^l \partial x^b}}\tilde{g}_{ij} + \cancel{\frac{\partial y^l}{\partial x^k}\frac{\partial y^j}{\partial x^b}\frac{\partial^2 y^i}{\partial y^l \partial x^a}}\tilde{g}_{ij} + \frac{\partial y^l}{\partial x^k}\frac{\partial y^i}{\partial x^a}\frac{\partial y^j}{\partial x^b}\frac{\partial \tilde{g}_{ij}}{\partial y^l} \\ & - \tilde{\Gamma}^n{}_{il}\frac{\partial y^i}{\partial x^a}\frac{\partial y^l}{\partial x^k}\tilde{g}_{nj}\frac{\partial y^j}{\partial x^b} - \cancel{\frac{\partial y^l}{\partial x^k}\frac{\partial y^j}{\partial x^b}\frac{\partial^2 y^i}{\partial y^l \partial x^a}}\tilde{g}_{ij} \\ & - \tilde{\Gamma}^n{}_{lj}\frac{\partial y^l}{\partial x^k}\frac{\partial y^j}{\partial x^b}\tilde{g}_{ni}\frac{\partial y^i}{\partial x^a} - \cancel{\frac{\partial y^l}{\partial x^k}\frac{\partial y^i}{\partial x^a}\frac{\partial^2 y^j}{\partial y^l \partial x^b}}\tilde{g}_{ij} \\ & \Bigg)\frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j}\frac{\partial x^k}{\partial y^l}\mathrm{d}y^i\mathrm{d}y^j\mathrm{d}y^l \\ = & \left(\frac{\partial \tilde{g}_{ij}}{\partial y^l} - \tilde{\Gamma}^n{}_{il}\tilde{g}_{nj} - \tilde{\Gamma}^n{}_{lj}\tilde{g}_{ni}\right)\cancel{\frac{\partial y^l}{\partial x^k}}\frac{\partial y^i}{\partial x^a}\cancel{\frac{\partial y^j}{\partial x^b}}\cancel{\frac{\partial x^a}{\partial y^i}}\frac{\partial x^b}{\partial y^j}\cancel{\frac{\partial x^k}{\partial y^l}}\mathrm{d}y^i\mathrm{d}y^j\mathrm{d}y^l \\ = & \tilde{\nabla}_l\,\tilde{g}_{ij}\,\mathrm{d}y^i\mathrm{d}y^j\mathrm{d}y^l = \tilde{\nabla}\tilde{g}\end{aligned}$$

4. Решение Правим смяна на променлива, от x към y, което означава че:

$$\Gamma^i{}_{jk} = \tilde{\Gamma}^a{}_{bc}\frac{\partial x^i}{\partial y^a}\frac{\partial y^b}{\partial x^j}\frac{\partial y^c}{\partial x^k} + \frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j \partial x^k} \qquad \frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^a}\frac{\partial y^a}{\partial x^i} \qquad \mathrm{d}x^a = \frac{\partial x^a}{\partial y^i}\mathrm{d}y^i$$

Тоест:

$$\begin{aligned}
&R^i{}_{jkl}\frac{\partial}{\partial x^i}\mathrm{d}x^j\mathrm{d}x^k\mathrm{d}x^l = \left(\frac{\partial}{\partial x^l}\Gamma^i{}_{jk} - \frac{\partial}{\partial x^k}\Gamma^i{}_{jl} + \Gamma^i{}_{ml}\Gamma^m{}_{jk} - \Gamma^i{}_{mk}\Gamma^m{}_{jl}\right)\frac{\partial}{\partial x^i}\mathrm{d}x^j\mathrm{d}x^k\mathrm{d}x^l \\
&= \left(\frac{\partial}{\partial y^d}\frac{\partial y^d}{\partial x^l}\left(\tilde{\Gamma}^a{}_{bc}\frac{\partial x^i}{\partial y^a}\frac{\partial y^b}{\partial x^j}\frac{\partial y^c}{\partial x^k} + \frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j\partial x^k}\right) - \frac{\partial}{\partial y^c}\frac{\partial y^c}{\partial x^k}\left(\tilde{\Gamma}^a{}_{bd}\frac{\partial x^i}{\partial y^a}\frac{\partial y^b}{\partial x^j}\frac{\partial y^d}{\partial x^l} + \frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j\partial x^l}\right)\right. \\
&\quad + \left(\tilde{\Gamma}^a{}_{nd}\frac{\partial x^i}{\partial y^a}\frac{\partial y^n}{\partial x^m}\frac{\partial y^d}{\partial x^l} + \frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^m\partial x^l}\right)\left(\tilde{\Gamma}^n{}_{bc}\frac{\partial x^m}{\partial y^n}\frac{\partial y^b}{\partial x^j}\frac{\partial y^c}{\partial x^k} + \frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j\partial x^k}\right) \\
&\quad - \left(\tilde{\Gamma}^a{}_{nc}\frac{\partial x^i}{\partial y^a}\frac{\partial y^n}{\partial x^m}\frac{\partial y^c}{\partial x^k} + \frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^m\partial x^k}\right)\left(\tilde{\Gamma}^n{}_{bd}\frac{\partial x^m}{\partial y^n}\frac{\partial y^b}{\partial x^j}\frac{\partial y^d}{\partial x^l} + \frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j\partial x^l}\right) \\
&\quad \left.\right)\frac{\partial}{\partial y^a}\frac{\partial y^a}{\partial x^i}\frac{\partial x^j}{\partial y^b}\mathrm{d}y^b\frac{\partial x^k}{\partial y^c}\mathrm{d}y^c\frac{\partial x^l}{\partial y^d}\mathrm{d}y^d \\
&= \left(\frac{\partial}{\partial y^d}\tilde{\Gamma}^a{}_{bc}\left(\frac{\partial y^d}{\partial x^l}\frac{\partial x^i}{\partial y^a}\frac{\partial y^b}{\partial x^j}\frac{\partial y^c}{\partial x^k}\right) + \cancel{\frac{\partial y^d}{\partial x^l}\frac{\partial}{\partial y^d}\frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j\partial x^k}}\right. \\
&\quad - \frac{\partial}{\partial y^c}\tilde{\Gamma}^a{}_{bd}\left(\frac{\partial y^c}{\partial x^k}\frac{\partial x^i}{\partial y^a}\frac{\partial y^b}{\partial x^j}\frac{\partial y^d}{\partial x^l}\right) - \cancel{\frac{\partial y^c}{\partial x^k}\frac{\partial}{\partial y^c}\frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j\partial x^l}} \\
&\quad + \tilde{\Gamma}^a{}_{nd}\frac{\partial x^i}{\partial y^a}\cancel{\frac{\partial y^n}{\partial x^m}}\frac{\partial y^d}{\partial x^l}\tilde{\Gamma}^n{}_{bc}\cancel{\frac{\partial x^m}{\partial y^n}}\frac{\partial y^b}{\partial x^j}\frac{\partial y^c}{\partial x^k} + \tilde{\Gamma}^a{}_{nd}\frac{\partial x^i}{\partial y^a}\frac{\partial y^n}{\partial x^m}\frac{\partial y^d}{\partial x^l}\frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j\partial x^k} \\
&\quad + \tilde{\Gamma}^n{}_{bc}\frac{\partial x^m}{\partial y^n}\frac{\partial y^b}{\partial x^j}\frac{\partial y^c}{\partial x^k}\frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^m\partial x^l} + \frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^m\partial x^l}\frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j\partial x^k} \\
&\quad - \tilde{\Gamma}^a{}_{nc}\frac{\partial x^i}{\partial y^a}\cancel{\frac{\partial y^n}{\partial x^m}}\frac{\partial y^c}{\partial x^k}\tilde{\Gamma}^n{}_{bd}\cancel{\frac{\partial x^m}{\partial y^n}}\frac{\partial y^b}{\partial x^j}\frac{\partial y^d}{\partial x^l} - \tilde{\Gamma}^a{}_{nc}\frac{\partial x^i}{\partial y^a}\frac{\partial y^n}{\partial x^m}\frac{\partial y^c}{\partial x^k}\frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j\partial x^l} \\
&\quad - \tilde{\Gamma}^n{}_{bd}\frac{\partial x^m}{\partial y^n}\frac{\partial y^b}{\partial x^j}\frac{\partial y^d}{\partial x^l}\frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^m\partial x^k} - \frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^m\partial x^k}\frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j\partial x^l} \\
&\quad \left.\right)\frac{\partial y^a}{\partial x^i}\frac{\partial x^j}{\partial y^b}\frac{\partial x^k}{\partial y^c}\frac{\partial x^l}{\partial y^d}\frac{\partial}{\partial y^a}\mathrm{d}y^b\mathrm{d}y^c\mathrm{d}y^d
\end{aligned}$$

$\frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^m\partial x^l}\frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j\partial x^k} = \frac{\partial x^i}{\partial x^m}\frac{\partial^2 y^p}{\partial y^p\partial y^p}\frac{\partial x^m}{\partial x^l}\frac{\partial^2 x^l}{\partial x^j\partial x^k} = 0$ Аналогично $\frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^m\partial x^k}\frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j\partial x^l} = 0$

По формулата виждаме, че $\tilde{\Gamma}^a{}_{nd} = \tilde{\Gamma}^a{}_{nc}\cancel{\frac{\partial y^a}{\partial y^a}}\cancel{\frac{\partial y^n}{\partial y^n}}\frac{\partial y^c}{\partial y^d} + \frac{\partial y^a}{\partial y^p}\frac{\partial^2 y^p}{\partial y^n\partial y^d} = \tilde{\Gamma}^a{}_{nc}\frac{\partial y^c}{\partial y^d}$, следователно:

$$\tilde{\Gamma}^a{}_{nd}\frac{\partial x^i}{\partial y^a}\frac{\partial y^n}{\partial x^m}\frac{\partial y^d}{\partial x^l}\frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j\partial x^k} - \tilde{\Gamma}^a{}_{nc}\frac{\partial x^i}{\partial y^a}\frac{\partial y^n}{\partial x^m}\frac{\partial y^c}{\partial x^k}\frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j\partial x^l} =$$

$$\tilde{\Gamma}^a{}_{nc}\cancel{\frac{\partial y^c}{\partial y^d}}\frac{\partial x^i}{\partial y^a}\frac{\partial y^n}{\partial x^m}\cancel{\frac{\partial y^d}{\partial x^l}}\frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j\partial x^k} - \tilde{\Gamma}^a{}_{nc}\frac{\partial x^i}{\partial y^a}\frac{\partial y^n}{\partial x^m}\frac{\partial y^c}{\partial x^k}\frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j\partial x^l} = 0$$

Аналогично $\tilde{\Gamma}^n{}_{bc}\frac{\partial x^m}{\partial y^n}\frac{\partial y^b}{\partial x^j}\frac{\partial y^c}{\partial x^k}\frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^m\partial x^l} - \tilde{\Gamma}^n{}_{bd}\frac{\partial x^m}{\partial y^n}\frac{\partial y^b}{\partial x^j}\frac{\partial y^d}{\partial x^l}\frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^m\partial x^k} = 0$

Следователно:

$$\begin{aligned}
&R^i{}_{jkl}\frac{\partial}{\partial x^i}\mathrm{d}x^j\mathrm{d}x^k\mathrm{d}x^l \\
&= \left(\frac{\partial}{\partial y^d}\tilde{\Gamma}^a{}_{bc} - \frac{\partial}{\partial y^c}\tilde{\Gamma}^a{}_{bd} + \tilde{\Gamma}^a{}_{nd}\tilde{\Gamma}^n{}_{bc} - \tilde{\Gamma}^a{}_{nc}\tilde{\Gamma}^n{}_{bd}\right)\cancel{\frac{\partial y^d}{\partial x^l}}\cancel{\frac{\partial x^i}{\partial y^a}}\cancel{\frac{\partial y^b}{\partial x^j}}\cancel{\frac{\partial y^c}{\partial x^k}}\cancel{\frac{\partial y^a}{\partial x^i}}\cancel{\frac{\partial x^j}{\partial y^b}}\cancel{\frac{\partial x^k}{\partial y^c}}\cancel{\frac{\partial x^l}{\partial y^d}}\frac{\partial}{\partial y^a}\mathrm{d}y^b\mathrm{d}y^c\mathrm{d}y^d \\
&= \tilde{R}^a{}_{bcd}\frac{\partial}{\partial y^a}\mathrm{d}y^b\mathrm{d}y^c\mathrm{d}y^d
\end{aligned}$$

Бонус задача Докажете Тъждеството на Бианки

Решение

$$\begin{aligned} R^i{}_{abc} + R^i{}_{bca} + R^i{}_{cab} = &\frac{\partial}{\partial x^c}\Gamma^i{}_{ab} - \frac{\partial}{\partial x^b}\Gamma^i{}_{ac} \quad + \Gamma^i{}_{mc}\Gamma^m{}_{ab} - \Gamma^i{}_{mb}\Gamma^m{}_{ac} \\ &+ \frac{\partial}{\partial x^a}\Gamma^i{}_{bc} - \frac{\partial}{\partial x^c}\Gamma^i{}_{ba} \quad + \Gamma^i{}_{ma}\Gamma^m{}_{bc} - \Gamma^i{}_{mc}\Gamma^m{}_{ba} \\ &+ \frac{\partial}{\partial x^b}\Gamma^i{}_{ca} - \frac{\partial}{\partial x^a}\Gamma^i{}_{cb} \quad + \Gamma^i{}_{mb}\Gamma^m{}_{ca} - \Gamma^i{}_{ma}\Gamma^m{}_{cb} \end{aligned}$$

Използвайки формулата за смяна на променливи при кривини, виждаме че:

$$\Gamma^i{}_{ab} = \Gamma^i{}_{ba}\frac{\partial y^i}{\partial y^i}\frac{\partial y^b}{\partial y^a}\frac{\partial y^a}{\partial y^b} + \frac{\partial y^i}{\partial y^p}\frac{\partial^2 y^p}{\partial y^a \partial y^b} = \Gamma^i{}_{ba}$$

Тоест:

$$\begin{aligned} &\frac{\partial}{\partial x^c}\Gamma^i{}_{ab} - \frac{\partial}{\partial x^c}\Gamma^i{}_{ba} = \frac{\partial}{\partial x^c}\Gamma^i{}_{ab} - \frac{\partial}{\partial x^c}\Gamma^i{}_{ab} = 0 \\ &\text{Аналогично} \quad \frac{\partial}{\partial x^b}\Gamma^i{}_{ca} - \frac{\partial}{\partial x^b}\Gamma^i{}_{ac} = 0 \quad \frac{\partial}{\partial x^a}\Gamma^i{}_{bc} - \frac{\partial}{\partial x^a}\Gamma^i{}_{cb} = 0 \\ &\Gamma^i{}_{mc}\Gamma^m{}_{ab} - \Gamma^i{}_{mc}\Gamma^m{}_{ba} = \Gamma^i{}_{mc}\Gamma^m{}_{ab} - \Gamma^i{}_{mc}\Gamma^m{}_{ab} = 0 \\ &\text{Аналогично} \quad \Gamma^i{}_{mb}\Gamma^m{}_{ca} - \Gamma^i{}_{mb}\Gamma^m{}_{ac} = 0 \quad \Gamma^i{}_{ma}\Gamma^m{}_{bc} - \Gamma^i{}_{ma}\Gamma^m{}_{cb} = 0 \end{aligned}$$

Следователно $R^i{}_{abc} + R^i{}_{bca} + R^i{}_{cab} = 0$

6. Решение 0MI0800028, следователно ще разгледаме отместването на перихелия на Нептун. Ще използваме, че:

- гравитационния радиус на слънцето е $\alpha_{\text{Слънце}} = 2.953$ km
- средния радиус на Нептун е $r_{\text{средно,Нептун}} = 4.5 \cdot 10^9$ km
- ексцентритетът на Нептун е $e = 0.008678$
- Нептун прави една обиколка (около Слънцето) на всеки 164.8 години

По формулата на Айнщайн:

$$\begin{aligned} \Delta\varphi &\approx \frac{3\pi\alpha_{\text{Слънце}}}{r_{\text{средно,Нептун}}(1-e^2)} \approx \frac{3\pi \cdot 2.953}{4.5 \cdot 10^9 \cdot (1 - 0.008678^2)} \approx \pi\frac{8859 \cdot 10^{-3}}{4.5 \cdot 10^9 \cdot (1 - 0.000075307684)} \\ &\approx \pi\frac{8859}{4.5 \cdot 10^{12} \cdot 0.999924692316} \approx 6.18521453154852035278818 59... \cdot 10^{-9} \text{ радиани на обиколка} \end{aligned}$$

Превръщайки към дъгови секунди на 100 години :

$$\begin{aligned} &\approx 0.00000000618521453154852035278818 59... \cdot \frac{180}{\pi}3600 \cdot \frac{100}{164.8} \\ &\approx 0.00077414567775884732...'' \end{aligned}$$

По точната формула:

$$\Delta\varphi = T_{\text{real}} - 2\pi = \frac{2\pi}{M\left(\sqrt{1 - \frac{\alpha_{\text{Слънце}}}{r_{\min}} - \frac{2\alpha_{\text{Слънце}}}{r_{\max}}}, \sqrt{1 - \frac{2\alpha_{\text{Слънце}}}{r_{\min}} - \frac{\alpha_{\text{Слънце}}}{r_{\max}}}\right)} - 2\pi$$

$$r_{\min} = (1-e)r_{\text{средно,Нептун}} = (1-0.008678)\cdot 4.5\cdot 10^9 = 0.991322\cdot 4.5\cdot 10^9 = 4.460949\cdot 10^9$$
$$r_{\max} = (1+e)r_{\text{средно,Нептун}} = (1+0.008678)\cdot 4.5\cdot 10^9 = 1.008678\cdot 4.5\cdot 10^9 = 4.539051\cdot 10^9$$
$$\frac{\alpha_{\text{Слънце}}}{r_{\min}} = \frac{2.953}{4.460949\cdot 10^9} = \frac{2953}{4460949\cdot 10^3} \qquad \frac{\alpha_{\text{Слънце}}}{r_{\max}} = \frac{2.953}{4.539051\cdot 10^9} = \frac{2953}{4539051\cdot 10^3}$$

$$\begin{aligned}\sqrt{1-\frac{\alpha_{\text{Слънце}}}{r_{\min}}-\frac{2\alpha_{\text{Слънце}}}{r_{\max}}} &= \sqrt{1-\frac{2953}{4460949\cdot 10^3}-2\frac{2953}{4539051\cdot 10^3}}\\ &= \sqrt{\frac{4460949\cdot 10^3\cdot 4539051\cdot 10^3 - 2953\cdot 4539051 - 2\cdot 2953\cdot 4460949}{4460949\cdot 10^3\cdot 4539051\cdot 10^3}}\\ &= \sqrt{\frac{1}{4460949\cdot 10^3\cdot 4539051\cdot 10^3}}\sqrt{20248474979648817603}\end{aligned}$$

$$\begin{aligned}\sqrt{1-\frac{2\alpha_{\text{Слънце}}}{r_{\min}}-\frac{\alpha_{\text{Слънце}}}{r_{\max}}} &= \sqrt{1-2\frac{2953}{4460949\cdot 10^3}-\frac{2953}{4539051\cdot 10^3}}\\ &= \sqrt{\frac{4460949\cdot 10^3\cdot 4539051\cdot 10^3 - 2\cdot 2953\cdot 4539051 - 2953\cdot 4460949}{4460949\cdot 10^3\cdot 4539051\cdot 10^3}}\\ &= \sqrt{\frac{1}{4460949\cdot 10^3\cdot 4539051\cdot 10^3}}\sqrt{20248474979418182397}\end{aligned}$$

$$\begin{aligned}&M\left(\sqrt{1-\frac{\alpha_{\text{Слънце}}}{r_{\min}}-\frac{2\alpha_{\text{Слънце}}}{r_{\max}}}, \sqrt{1-\frac{2\alpha_{\text{Слънце}}}{r_{\min}}-\frac{\alpha_{\text{Слънце}}}{r_{\max}}}\right)\\ &= M\left(\sqrt{\frac{1}{4460949\cdot 10^3\cdot 4539051\cdot 10^3}}\sqrt{20248474979648817603}\,,\right.\\ &\qquad \left.\sqrt{\frac{1}{4460949\cdot 10^3\cdot 4539051\cdot 10^3}}\sqrt{20248474979418182397}\right)\\ &= \sqrt{\frac{1}{4460949\cdot 4539051\cdot 10^6}}M\left(\sqrt{20248474979648817603}, \sqrt{20248474979418182397}\right)\\ &= 2.22230590215300266664 89...\cdot 10^{-10}\cdot 4.49983055009113648734 96...\cdot 10^9\\ &= 0.99999999901559253273566948868277025406747554 4... = M_{\text{стойност}}\end{aligned}$$

$$\frac{2\pi}{M_{\text{стойност}}} = 6.28318531336480101760716936984332803180490350...$$

$$\begin{aligned}\Delta\varphi &= 6.28318531336480101760716936984332803180490350... - 2\pi\\ &= 6.18521454068188260328432 22...\cdot 10^{-9} \text{ радиани на обиколка}\\ &\text{Превръщайки към дъгови секунди на 100 години :}\\ &= 0.00000000618521454068188260328432 22...\cdot \frac{180}{\pi}3600\cdot\frac{100}{164.8}\\ &= 0.00077414567890198518...''\end{aligned}$$

Приближението на Айнщайн е достатъчно точно за 11 цифри след запетаята (различаващите цифри са удебелени и подчертани):

По Айнщайн: $\Delta\varphi \approx 0.0007741456\underline{\mathbf{7758847 32}}...''$

По точната: $\Delta\varphi = 0.0007741456\underline{\mathbf{8901985 18}}...''$